Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: Solutions

Unique ID: ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________________

April 10, 2026

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the matrices A, Q, and R and the vector $\boldsymbol{b}$ given by

$$A = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 & -1 & 6 \\ -9 & 7 & -22 \\ 0 & -2 & 2 \\ -6 & -4 & 8 \end{bmatrix} \qquad Q = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 & 0 & 2 \\ -3 & -2 & 0 \\ 0 & 1 & -3 \\ -2 & 3 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{b} = 3\sqrt{14}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

It is known that $A = QR$ and that Q has orthonormal columns.

Do not ignore the factor of $1/\sqrt{14}$ used to define the matrices A and Q and the factor of $3\sqrt{14}$ to define the vector $\boldsymbol{b}$!

(8 pts) (a) The $(3,1)$ *cofactor* of R is ___3___ and $\det(Q^\intercal Q) =$ ___1___.

(4 pts) (b) The equation $R\,\operatorname{adj}(R) = c \cdot I_3$ is valid for $c =$ ___-6___.

(10 pts) (c) Find the least squares solution $\widehat{\boldsymbol{x}}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit. Fill in the blank vector below for clarity.

Solution. From class we know the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ reduces to $R\widehat{\boldsymbol{x}} = Q^\intercal \boldsymbol{b}$. The relevant data here is then

$$\frac{1}{\sqrt{14}}\overset{Q^\intercal}{\begin{bmatrix} 1 & -3 & 0 & -2 \\ 0 & -2 & 1 & 3 \\ 2 & 0 & -3 & 1 \end{bmatrix}} \overset{\boldsymbol{b}}{3\sqrt{14}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 0 \\ -12 \\ -6 \end{bmatrix} \qquad \begin{aligned} 3\,\widehat{x}_1 - \widehat{x}_2 + 4\,\widehat{x}_3 &= 0 & \to \widehat{x}_1 &= \tfrac{-9-4(-6)}{3} = 5 \\ -2\,\widehat{x}_2 + 5\,\widehat{x}_3 &= -12 & \to \widehat{x}_2 &= \tfrac{-12-5(-6)}{-2} = -9 \\ \widehat{x}_3 &= -6 & \to \widehat{x}_3 &= -6 \end{aligned}$$

This gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 5 & -9 & -6 \end{bmatrix}^\intercal$.

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ -9 \\ -6 \end{bmatrix}$$

(8 pts) (d) Let M be the matrix given by $\overset{M}{\begin{bmatrix} -207 & 44 & -5 \\ -979 & 208 & -23 \\ -111 & 23 & 1 \end{bmatrix}} = \overset{X}{\begin{bmatrix} 1 & -4 & 4 \\ 5 & -19 & 19 \\ 2 & -3 & 4 \end{bmatrix}} \overset{R}{\begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -19 & 4 & 0 \\ 18 & -4 & 1 \\ 23 & -5 & 1 \end{bmatrix}}$. Calculate $\chi_M(4)$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

Solution. The given equation expresses M as *similar* to R. From class, we then know that $\chi_M(t) = \chi_R(t)$. Since R is upper-triangular, we easily calculate

$$\chi_R(t) = (t-3)\cdot(t+2)\cdot(t-1)$$

It follows that $\chi_M(4) = \chi_R(4) = (4-3)\cdot(4+2)\cdot(4-1) = 18$.

$\chi_M(4) =$ ___18___

(10 pts) **Problem 2.** Let K be an $n \times n$ matrix satisfying $K^\intercal = c \cdot K$ where c is a real scalar. Show that $\chi_K(ct) = c^n \cdot \chi_K(t)$. Clearly explain your reasoning and avoid circular logic to receive credit.

Hint. Recall that every square matrix M satisfies $\det(M) = \det(M^\intercal)$.

Solution. We are given that $K^\intercal = c \cdot K$. We wish to demonstrate that $\chi_K(ct) = c^n \cdot \chi_K(t)$. To do so, note that

$$\begin{aligned}
\chi_K(ct) &= \det(ct \cdot I_n - K) \\
&= \det\big((ct \cdot I_n - K)^\intercal\big) \\
&= \det(ct \cdot I_n^\intercal - K^\intercal) \\
&= \det(ct \cdot I_n - c \cdot K) \\
&= \det\big(c(t \cdot I_n - K)\big) \\
&= c^n \cdot \det(t \cdot I_n - K) \\
&= c^n \cdot \chi_K(t)
\end{aligned}$$

Here the second equality follows from the hint that $\det(M) = \det(M^\intercal)$ for any square M. The penultimate equality follows from the fact that $\det(c \cdot M) = c^n \cdot \det(M)$, which follows from the fact that scaling one row of M by c scales the determinant by c and therefore scaling all n rows by c scales the determinant by c^n.

(10 pts) **Problem 3.** Clearly explain whether or not $A = \begin{bmatrix} 9 & 1 \\ -4 & 5 \end{bmatrix}$ is diagonalizable. Select your conclusion below for clarity.

Solution. The characteristic polynomial of A is

$$\chi_A(t) = t^2 - \operatorname{trace}(A)\, t + \det(A) = t^2 - 14t + 49 = (t-7)^2$$

The only eigenvalue of A is $\lambda = 7$. The only eigenspace is

$$\mathcal{E}_A(7) = \operatorname{Null} \overbrace{\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}}^{7 \cdot I_2 - A} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

The dimension of the only eigenspace of this 2×2 matrix is one, so A is *not diagonalizable*.

○ A *is* diagonalizable   √ A is *not* diagonalizable

Problem 4. The data below depicts a ***unitary*** 3×3 matrix U, a diagonal 3×3 matrix D, and a vector $\boldsymbol{v}\in\mathbb{C}^3$.

$$U=\frac{1}{2}\begin{bmatrix}1 & -i-1 & -i\\ -i & -i+1 & -1\\ -i+1 & 0 & i+1\end{bmatrix}\qquad D=\begin{bmatrix}-1 & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & 1\end{bmatrix}\qquad \boldsymbol{v}=\begin{bmatrix}1\\1\\i\end{bmatrix}$$

Throughout this problem, let $A=UDU^{-1}$. ***Do not ignore the factor of*** $^1/_2$ ***used to define*** U***!***

(4 pts) (*a*) The coefficient of t^2 in $\chi_A(t)$ is $\underline{\quad i \quad}$ and the constant coefficient of $\chi_A(t)$ is $\underline{\quad -i \quad}$.

(4 pts) (*b*) Only one of the following statements accurately applies to the matrix U. Select this statement.

○ $U^2=U$ ○ $U^*=U$ ✓ $\chi_U(0)\neq 0$ ○ $U^2=I_3$ ○ $\exp(U)=I_3$

(4 pts) (*c*) Only one of the following statements accurately applies to the matrix A. Select this statement.

✓ A is not real-symmetric and also not Hermitian ○ A is not real-symmetric but is Hermitian

○ A is real-symmetric but not Hermitian ○ A is real-symmetric and also Hermitian

(8 pts) (*d*) Calculate $\langle \boldsymbol{w},\boldsymbol{v}\rangle$ where $\boldsymbol{w}=\frac{1}{2}\cdot\begin{bmatrix}-i+1 & 0 & i+1\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

Solution.

$$\begin{aligned}
\langle \boldsymbol{w},\boldsymbol{v}\rangle &= \left\langle \frac{1}{2}\begin{bmatrix}-i+1 & 0 & i+1\end{bmatrix}^\intercal, \begin{bmatrix}1 & 1 & i\end{bmatrix}^\intercal\right\rangle\\
&= \frac{1}{2}\langle \begin{bmatrix}-i+1 & 0 & i+1\end{bmatrix}^\intercal, \begin{bmatrix}1 & 1 & i\end{bmatrix}^\intercal\rangle\\
&= \frac{1}{2}\left\{\overline{(-i+1)}\cdot(1)+\overline{(0)}\cdot(1)+\overline{(i+1)}\cdot(i)\right\}\\
&= \frac{1}{2}\{(i+1)\cdot(1)+(-i+1)\cdot(i)\}\\
&= \frac{1}{2}\{i+1+i+1\}\\
&= i+1
\end{aligned}$$

$\langle \boldsymbol{w},\boldsymbol{v}\rangle = \underline{\quad i+1 \quad}$

(10 pts) (*e*) Calculate $A^2\boldsymbol{v}$. Clearly explain your reasoning to receive credit. Fill in the blank vector below for clarity.
Hint. You may use the calculation $U^*\boldsymbol{v}=\begin{bmatrix}i & i & i\end{bmatrix}^\intercal$.

Solution. Since U is unitary and since $A=UDU^{-1}$, we know that $A^2=UD^2U^*$. It follows that

$$\begin{aligned}
A^2\boldsymbol{v} &= \overbrace{\frac{1}{2}\begin{bmatrix}1 & -i-1 & -i\\ -i & -i+1 & -1\\ -i+1 & 0 & i+1\end{bmatrix}}^{U}\overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}}^{D^2}\overbrace{\frac{1}{2}\begin{bmatrix}1 & i & i+1\\ i-1 & i+1 & 0\\ i & -1 & -i+1\end{bmatrix}}^{U^*}\overbrace{\begin{bmatrix}1\\1\\i\end{bmatrix}}^{\boldsymbol{v}}\\
&= \frac{1}{2}\begin{bmatrix}1 & -i-1 & -i\\ -i & -i+1 & -1\\ -i+1 & 0 & i+1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}i\\i\\i\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}1 & -i-1 & -i\\ -i & -i+1 & -1\\ -i+1 & 0 & i+1\end{bmatrix}\begin{bmatrix}i\\-i\\i\end{bmatrix}\\
&= \frac{i}{2}\begin{bmatrix}1 & -i-1 & -i\\ -i & -i+1 & -1\\ -i+1 & 0 & i+1\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix}\\
&= \frac{i}{2}\begin{bmatrix}2\\-2\\2\end{bmatrix}
\end{aligned}$$

$$A^2\boldsymbol{v}=\begin{bmatrix}i\\-i\\i\end{bmatrix}$$

Problem 5. Consider the spectral factorization $S = UDU^\intercal$ where S is the real-symmetric matrix, U is the unitary matrix, and D is the real-diagonal matrix given by

$$S = \begin{bmatrix} 11/10 & 11/10 & -1/5 & -1/5 \\ 11/10 & 11/10 & -1/5 & -1/5 \\ -1/5 & -1/5 & 7/5 & 7/5 \\ -1/5 & -1/5 & 7/5 & 7/5 \end{bmatrix} \qquad U = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 & 1 & -1 \\ -2 & 2 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ -1 & 1 & -2 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $q(x_1, x_2, x_3, x_4)$ be the quadratic form defined by $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$. ***Do not ignore the factor of*** $1/\sqrt{10}$ ***used to define*** U***!***

(4 pts) (a) Which categories of definiteness apply to S? Select all that apply (no partial credit here).

○ positive definite ✓ positive semidefinite ○ negative definite ○ negative semidefinite ○ indefinite

(6 pts) (b) The coefficient of x_2^2 in $q(\boldsymbol{x})$ is ___11/10___ and the coefficient of x_2x_4 in $q(\boldsymbol{x})$ is ___−2/5___.

(10 pts) (c) Use the technique of "completing the square" to calculate the value of $q(x_1, x_2, x_3, x_4)$ when

$$x_1 = 510 \qquad x_2 = -500 \qquad x_3 = -237 \qquad x_4 = 237$$

Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

Solution. The given spectral factorization allows us to quickly "complete the square" and write

$$\begin{aligned} q(\boldsymbol{x}) &= 0 \cdot y_1^2 + 2 \cdot y_2^2 + 3 \cdot y_3^2 + 0 \cdot y_4^2 \\ &= 2 \cdot y_2^2 + 3 \cdot y_3^2 \end{aligned}$$

where $\boldsymbol{y} = U^\intercal \boldsymbol{x}$. The values of y_1 and y_4 are irrelevant, so we need only compute

$$y_2 = \frac{2\,x_1 + 2\,x_2 + x_3 + x_4}{\sqrt{10}} \qquad y_3 = \frac{x_1 + x_2 - 2\,x_3 - 2\,x_4}{\sqrt{10}}$$

At the specified values of x_1, x_2, x_3, x_4 we then have

$$y_2 = \frac{20}{\sqrt{10}} = 2\sqrt{10} \qquad y_3 = \frac{10}{\sqrt{10}} = \sqrt{10}$$

Putting everything together, we have

$$\begin{aligned} q(510, -500, -237, 237) &= 2 \cdot y_2^2 + 3 \cdot y_3^2 \\ &= 2 \cdot (2\sqrt{10})^2 + 3 \cdot (\sqrt{10})^2 \\ &= 2 \cdot (40) + 3 \cdot (10) \\ &= 110 \end{aligned}$$

$q(510, -500, -237, 237) =$ ___110___