Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

April 10, 2026

- There are 100 points and 5 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the matrices A, Q, and R and the vector $\boldsymbol{b}$ given by

$$A = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 & -1 & 6 \\ -9 & 7 & -22 \\ 0 & -2 & 2 \\ -6 & -4 & 8 \end{bmatrix} \qquad Q = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 & 0 & 2 \\ -3 & -2 & 0 \\ 0 & 1 & -3 \\ -2 & 3 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{b} = 3\sqrt{14}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

It is known that $A = QR$ and that Q has orthonormal columns.

Do not ignore the factor of $1/\sqrt{14}$ ***used to define the matrices*** A ***and*** Q ***and the factor of*** $3\sqrt{14}$ ***to define the vector*** $\boldsymbol{b}$***!***

(8 pts) (a) The $(3,1)$ *cofactor* of R is ________ and $\det(Q^\intercal Q) =$ ________.

(4 pts) (b) The equation $R\,\mathrm{adj}(R) = c \cdot I_3$ is valid for $c =$ ________.

(10 pts) (c) Find the least squares solution $\widehat{\boldsymbol{x}}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your reasoning to receive credit. Fill in the blank vector below for clarity.

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

(8 pts) (d) Let M be the matrix given by $\overset{M}{\begin{bmatrix} -207 & 44 & -5 \\ -979 & 208 & -23 \\ -111 & 23 & 1 \end{bmatrix}} = \overset{X}{\begin{bmatrix} 1 & -4 & 4 \\ 5 & -19 & 19 \\ 2 & -3 & 4 \end{bmatrix}} \overset{R}{\begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -19 & 4 & 0 \\ 18 & -4 & 1 \\ 23 & -5 & 1 \end{bmatrix}}$. Calculate $\chi_M(4)$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

$\chi_M(4) =$ ________

(10 pts) **Problem 2.** Let K be an $n \times n$ matrix satisfying $K^\intercal = c \cdot K$ where c is a real scalar. Show that $\chi_K(ct) = c^n \cdot \chi_K(t)$. Clearly explain your reasoning and avoid circular logic to receive credit.

Hint. Recall that every square matrix M satisfies $\det(M) = \det(M^\intercal)$.

(10 pts) **Problem 3.** Clearly explain whether or not $A = \begin{bmatrix} 9 & 1 \\ -4 & 5 \end{bmatrix}$ is diagonalizable. Select your conclusion below for clarity.

○ A *is* diagonalizable ○ A is *not* diagonalizable

Problem 4. The data below depicts a ***unitary*** 3×3 matrix U, a diagonal 3×3 matrix D, and a vector $\boldsymbol{v} \in \mathbb{C}^3$.

$$U = \frac{1}{2}\begin{bmatrix} 1 & -i-1 & -i \\ -i & -i+1 & -1 \\ -i+1 & 0 & i+1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix}$$

Throughout this problem, let $A = UDU^{-1}$. ***Do not ignore the factor of*** $^1/_2$ ***used to define*** U**!**

(4 pts) (*a*) The coefficient of t^2 in $\chi_A(t)$ is _______ and the constant coefficient of $\chi_A(t)$ is _______.

(4 pts) (*b*) Only one of the following statements accurately applies to the matrix U. Select this statement.

○ $U^2 = U$ ○ $U^* = U$ ○ $\chi_U(0) \neq 0$ ○ $U^2 = I_3$ ○ $\exp(U) = I_3$

(4 pts) (*c*) Only one of the following statements accurately applies to the matrix A. Select this statement.

○ A is not real-symmetric and also not Hermitian ○ A is not real-symmetric but is Hermitian

○ A is real-symmetric but not Hermitian ○ A is real-symmetric and also Hermitian

(8 pts) (*d*) Calculate $\langle \boldsymbol{w}, \boldsymbol{v} \rangle$ where $\boldsymbol{w} = \frac{1}{2} \cdot \begin{bmatrix} -i+1 & 0 & i+1 \end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

$\langle \boldsymbol{w}, \boldsymbol{v} \rangle =$ _______________

(10 pts) (*e*) Calculate $A^2\boldsymbol{v}$. Clearly explain your reasoning to receive credit. Fill in the blank vector below for clarity.
Hint. You may use the calculation $U^*\boldsymbol{v} = \begin{bmatrix} i & i & i \end{bmatrix}^\intercal$.

$A^2\boldsymbol{v} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$

Problem 5. Consider the spectral factorization $S = UDU^\intercal$ where S is the real-symmetric matrix, U is the unitary matrix, and D is the real-diagonal matrix given by

$$S = \begin{bmatrix} 11/10 & 11/10 & -1/5 & -1/5 \\ 11/10 & 11/10 & -1/5 & -1/5 \\ -1/5 & -1/5 & 7/5 & 7/5 \\ -1/5 & -1/5 & 7/5 & 7/5 \end{bmatrix} \qquad U = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 2 & 1 & -1 \\ -2 & 2 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ -1 & 1 & -2 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $q(x_1, x_2, x_3, x_4)$ be the quadratic form defined by $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x}\rangle$. ***Do not ignore the factor of*** $1/\sqrt{10}$ ***used to define*** U***!***

(4 pts) (a) Which categories of definiteness apply to S? Select all that apply (no partial credit here).

○ positive definite ○ positive semidefinite ○ negative definite ○ negative semidefinite ○ indefinite

(6 pts) (b) The coefficient of x_2^2 in $q(\boldsymbol{x})$ is __________ and the coefficient of x_2x_4 in $q(\boldsymbol{x})$ is __________.

(10 pts) (c) Use the technique of "completing the square" to calculate the value of $q(x_1, x_2, x_3, x_4)$ when

$$x_1 = 510 \qquad x_2 = -500 \qquad x_3 = -237 \qquad x_4 = 237$$

Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

$q(510, -500, -237, 237) =$ __________